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New applications of pseudoanalytic function theory to the Dirac equation

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Abstract
In the present work, we establish a simple relation between the Dirac equation with a scalar and an electromagnetic potential in a two-dimensional case and a pair of decoupled Vekua equations. In general, these Vekua equations are bicomplex. However, we show that the whole theory of pseudoanalytic functions without modifications can be applied to these equations under a certain nonrestrictive condition. As an example we formulate the similarity principle which is the central reason why a pseudoanalytic function and as a consequence a spinor field depending on two space variables share many of the properties of analytic functions. One of the surprising consequences of the established relation with pseudoanalytic functions consists in the following result. Consider the Dirac equation with a scalar potential depending on one variable with fixed energy and mass. In general, this equation cannot be solved explicitly even if one looks for wavefunctions of one variable. Nevertheless, for such Dirac equation, we obtain an algorithmically simple procedure for constructing in explicit form a complete system of exact solutions (depending on two variables). These solutions generalize the system of powers $1, z, z^2, \ldots$ in complex analysis and are called formal powers. With their aid any regular solution of the Dirac equation can be represented by its Taylor series in formal powers.

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1. Introduction

The Dirac equation with a fixed energy and the Vekua equation describing pseudoanalytic (= generalized analytic) functions are both first-order elliptic systems, and it would be quite natural to expect a deep interrelation between their theories especially in the case when all potentials and wavefunctions in the Dirac equation depend on two space variables only. Nevertheless, there is not much work done in this direction[1] due to the fact that any of the traditional matrix representations of the Dirac operator does not allow us to visualize a relation between the Dirac equation in the two-dimensional case and the Vekua equation. The Dirac equation is a system of four complex equations which does not decouple in a two-dimensional situation but decouples in the one-dimensional case only.

In the present work, we establish a simple relation between the Dirac equation with a scalar and an electromagnetic potential in a two-dimensional case from one side and a pair of decoupled Vekua equations from the other. As a first step, we use the matrix transformation proposed in [9] (see also [10, 14]) which allows us to rewrite the Dirac equation in a covariant form as a biquaternionic equation. This is not our aim to discuss here the advantages of our biquaternionic reformulation of the Dirac equation compared with its other representations (the interested reader can find some of the arguments in [10]). We point out only that our transformation is $\mathbb{C}$-linear as well as the resulting Dirac operator, which is not the case for a better known biquaternionic reformulation of the Dirac operator introduced by Lanczos in [15] (see [6, 10] for more references). Moreover, in the time-dependent case, with a vanishing electromagnetic potential, our Dirac operator is real quaternionic.

Here, we exploit another attractive facet of our biquaternionic Dirac equation. In the two-dimensional case, it decouples into two separate Vekua equations. In general, these Vekua equations are bicomplex. However, we show that the whole theory of pseudoanalytic functions without modifications can be applied to these equations under a certain nonrestrictive condition. As an example, we formulate the similarity principle which is the central reason why a pseudoanalytic function and as a consequence a spinor field depending on two space variables share many of the properties of analytic functions; e.g., they are either identically zero or have isolated zeros. In this way, more results of the theory developed in [19] and in posterior works (see, e.g., [2, 18]) can be applied to the two-dimensional Dirac equation with a scalar and an electromagnetic potential. Nevertheless, in the present work, we concentrate on another non-trivial and surprising consequence of the established relation with pseudoanalytic functions. Consider the Dirac equation with a scalar potential depending on one variable with fixed energy and mass. In general, this equation cannot be solved explicitly even if one looks for wavefunctions of one variable. Nonetheless, the result of this work is an algorithmically simple procedure for obtaining in explicit form a complete system of exact solutions depending on two variables for such Dirac equation. This system of solutions is a generalization of the system of powers $1, z, z^2, \ldots$ in complex analysis and as such they are not appropriate for studying the Dirac equation on the whole plane. However, the very fact that it is always possible to obtain explicitly a complete system of exact solutions of the Dirac equation with scalar potential of one variable as well as the hope to be able to obtain explicitly not only the generalizations of positive powers but also those of the negative ones makes in our opinion this approach attractive and promising. The system of exact solutions for the Dirac equation with a one-dimensional scalar potential is obtained due to the proposed reduction of the Dirac equation to Vekua equations and due to L Bers' theory of Taylor series in formal powers.

In section 2, we introduce notation. In section 3, we give the biquaternionic reformulation of the Dirac equation. Let us emphasize that our biquaternionic Dirac equation is completely equivalent to the 'traditional' Dirac equation written in γ-matrices, we have a simple matrix transformation giving us a relation between their solutions. In section 4, we show that in a two-dimensional situation the Dirac equation with a scalar and an electromagnetic potential

[1] We refer to the work [1] where the theory of pseudoanalytic functions was used in a way completely different from ours for studying the two-dimensional Dirac equation with a scalar or a pseudoscalar potential.

decouples into a pair of bicomplex Vekua equations. We establish that if one of the coefficients in such Vekua equation has not zeros and does not turn into a zero divisor at any point of the domain of interest, the solutions will not be zero divisors either, and the whole theory of generalized analytic functions without modifications is applicable to the bicomplex Vekua equation.

In section 5, we adapt some definitions and results from L Bers' theory to bicomplex pseudoanalytic functions. Section 6 is dedicated to a special class of Vekua equations which have been studied recently (see [11, 12]) due to their close relation to stationary Schrödinger equations. In section 7, we show that the Dirac equation with a scalar potential depending on one space variable can be represented as a Vekua equation from the special class mentioned above. Here, we should note that the case of the scalar potential is only an example. The same is true, for example, for the electric potential. To the Vekua equation we apply L Bers' procedure for constructing corresponding formal powers which as was mentioned above are exact solutions of the Vekua equation and generalize the system of analytic functions $1, z, z^2, \ldots$. With their aid any regular solution of the Vekua equation can be represented by its Taylor series in formal powers.

2. Preliminaries

We denote by $\mathbb{H}(\mathbb{C})$ the algebra of complex quaternions (= biquaternions). The elements of $\mathbb{H}(\mathbb{C})$ have the form $q = \sum_{k=0}^{3} q_k e_k$ where $\{q_k\} \subset \mathbb{C}$, e_0 is the unit and $\{e_k | k = 1, 2, 3\}$ are the standard quaternionic imaginary units.

We denote the imaginary unit in $\mathbb{C}$ by i as usual. By definition i commutes with e_k, $k = \overline{0, 3}$. We will also use the vector representation of $q \in \mathbb{H}(\mathbb{C})$: $q = \mathrm{Sc}(q) + \mathrm{Vec}(q)$, where $\mathrm{Sc}(q) = q_0$ and $\mathrm{Vec}(q) = \overrightarrow{q} = \sum_{k=1}^{3} q_k e_k$. The quaternionic conjugation is defined as follows: $\overline{q} = q_0 - \overrightarrow{q}$.

By M^p we denote the operator of multiplication by p from the right-hand side

$$M^p q = q \cdot p.$$

The interested reader can find more information on complex quaternions, e.g., in [10] or [14].

Let q be a complex quaternion-valued differentiable function of $\mathbf{x} = (x_1, x_2, x_3)$. Denote

$$Dq = \sum_{k=1}^{3} e_k \frac{\partial}{\partial x_k} q.$$

This operator is sometimes called the Moisil-Theodorescu operator or the Dirac operator but the truth is that it was introduced already by W R Hamilton himself and studied in a great number of works (see, e.g., [5, 7, 8, 10, 14]).

3. Quaternionic reformulation of the Dirac equation

Consider the Dirac operator with scalar and electromagnetic potentials

$$\mathbb{D} = \gamma_0 \partial_t + \sum_{k=1}^{3} \gamma_k \partial_k + \mathrm{i}\left(m + p_{\mathrm{el}}\gamma_0 + \sum_{k=1}^{3} A_k \gamma_k + p_{\mathrm{sc}} \right)$$

where γ_j, $j = 0, 1, 2, 3$ are usual γ-matrices (see, e.g., [4, 17]), $m \in \mathbb{R}$, p_{el}, A_k and p_{sc} are real-valued functions.

In [9], a simple matrix transformation was obtained which allows us to rewrite the classical Dirac equation in quaternionic terms.

Let us introduce an auxiliary notation $\widetilde{f} := f(t, x_1, x_2, -x_3)$. The domain $\widetilde{G}$ is assumed to be obtained from the domain $G \subset \mathbb{R}^4$ by the reflection $x_3 \to -x_3$. The transformation announced above we denote as $\mathcal{A}$ and define it in the following way. A function $\Phi : G \subset \mathbb{R}^4 \to \mathbb{C}^4$ is transformed into a function $F : \widetilde{G} \subset \mathbb{R}^4 \to \mathbb{H}(C)$ by the rule $F = \mathcal{A}[\Phi] := \frac{1}{2}(-(\widetilde{\Phi}_1 - \widetilde{\Phi}_2)e_0 + \mathrm{i}(\widetilde{\Phi}_0 - \widetilde{\Phi}_3)e_1 - (\widetilde{\Phi}_0 + \widetilde{\Phi}_3)e_2 + \mathrm{i}(\widetilde{\Phi}_1 + \widetilde{\Phi}_2)e_3)$.

The inverse transformation $\mathcal{A}^{-1}$ is defined as follows:

$$\Phi = \mathcal{A}^{-1}[F] = (-\mathrm{i}\widetilde{F}_1 - \widetilde{F}_2, -\widetilde{F}_0 - \mathrm{i}\widetilde{F}_3, \widetilde{F}_0 - \mathrm{i}\widetilde{F}_3, \mathrm{i}\widetilde{F}_1 - \widetilde{F}_2)^T.$$

Let us present the introduced transformations in a more explicit matrix form which relates the components of a $\mathbb{C}^4$-valued function Φ with the components of an $\mathbb{H}(\mathbb{C})$-valued function F:

$$F = \mathcal{A}[\Phi] = \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ \mathrm{i} & 0 & 0 & -\mathrm{i} \\ -1 & 0 & 0 & -1 \\ 0 & \mathrm{i} & \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} \widetilde{\Phi}_0 \\ \widetilde{\Phi}_1 \\ \widetilde{\Phi}_2 \\ \widetilde{\Phi}_3 \end{pmatrix}$$

and

$$\Phi = \mathcal{A}^{-1}[F] = \begin{pmatrix} 0 & -\mathrm{i} & -1 & 0 \\ -1 & 0 & 0 & -\mathrm{i} \\ 1 & 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & -1 & 0 \end{pmatrix}\begin{pmatrix} \widetilde{F}_0 \\ \widetilde{F}_1 \\ \widetilde{F}_2 \\ \widetilde{F}_3 \end{pmatrix}.$$

Denote

$$R = D - \partial_t M^{e_1} + \mathbf{a} + M^{-\mathrm{i}(\widetilde{p}_{\mathrm{el}} e_1 - \mathrm{i}(\widetilde{p}_{\mathrm{sc}} + m)e_2)}$$

where $\mathbf{a} = \mathrm{i}(\widetilde{A}_1 e_1 + \widetilde{A}_2 e_2 - \widetilde{A}_3 e_3)$. The following equality holds [10]:

$$R = \mathcal{A}\gamma_1\gamma_2\gamma_3\mathbb{D}\mathcal{A}^{-1}.$$

That is, a $\mathbb{C}^4$-valued function Φ is a solution of the equation

$$\mathbb{D}\Phi = 0 \qquad \text{in} \quad G$$

iff the complex quaternionic function $F = \mathcal{A}\Phi$ is a solution of the quaternionic equation

$$RF = 0 \qquad \text{in} \quad \widetilde{G}.$$

Note that in the absence of the electromagnetic potential the operator R becomes real quaternionic which is an important property (see [13]).

In what follows we assume that potentials are time independent and consider solutions with fixed energy: $\Phi(t, \mathbf{x}) = \Phi_\omega(\mathbf{x})\,\mathrm{e}^{\mathrm{i}\omega t}$. The equation for Φ_ω has the form

$$\mathbb{D}_\omega\Phi_\omega = 0 \qquad \text{in} \quad \widehat{G} \tag{1}$$

where $\widehat{G}$ is a domain in $\mathbb{R}^3$,

$$\mathbb{D}_\omega = \mathrm{i}\omega\gamma_0 + \sum_{k=1}^{3}\gamma_k\partial_k + \mathrm{i}\left(m + p_{\mathrm{el}}\gamma_0 + \sum_{k=1}^{3} A_k\gamma_k + p_{\mathrm{sc}}\right).$$

We have

$$R_\omega = \mathcal{A}\gamma_1\gamma_2\gamma_3\mathbb{D}_\omega\mathcal{A}^{-1},$$

where

$$R_\omega = D + \mathbf{a} + M^{\mathbf{b}}$$

with $\mathbf{b} = -\mathrm{i}((\widetilde{p}_{\mathrm{el}} + \omega)e_1 - \mathrm{i}(\widetilde{p}_{\mathrm{sc}} + m)e_2)$. Thus, equation (1) turns into the complex quaternionic equation

$$R_\omega q = 0 \tag{2}$$

where q is a complex quaternion-valued function.

4. The Dirac equation in a two-dimensional case as a bicomplex Vekua equation

Let us introduce the following notation. For any complex quaternion q we denote by Q_1 and Q_2 its bicomplex components:

$$Q_1 = q_0 + q_3 e_3 \qquad \text{and} \qquad Q_2 = q_2 - q_1 e_3.$$

Then q can be represented as follows: $q = Q_1 + Q_2 e_2$. For the operator D, we have $D = D_1 + D_2 e_2$ with $D_1 = e_3 \partial_3$ and $D_2 = \partial_2 - \partial_1 e_3$. Note that $\mathbf{b} = B e_2$ with $B = -(\widetilde{p}_{\mathrm{sc}} + m) + \mathrm{i}(\widetilde{p}_{\mathrm{el}} + \omega) e_3$, $\mathbf{a} = A_1 + A_2 e_2$ with $A_1 = a_3 e_3$ and $A_2 = a_2 - a_1 e_3$.

We obtain that equation (2) is equivalent to the system

$$D_1 Q_1 - D_2 \overline{Q_2} + A_1 Q_1 - A_2 \overline{Q_2} - \overline{B} Q_2 = 0, \tag{3}$$

$$D_2 \overline{Q_1} + D_1 Q_2 + A_2 \overline{Q_1} + A_1 Q_2 + B Q_1 = 0, \tag{4}$$

where Q_1 and Q_2 are bicomplex components of q. We stress that the system (3), (4) is equivalent to the Dirac equation in γ-matrices (1).

Let us suppose all fields in our model to be independent of x_3 and $A_1 = a_3 e_3 \equiv 0$. Then the system (3), (4) decouples, and we obtain two separate bicomplex equations

$$\overline{D}_2 Q_2 = -\overline{A}_2 Q_2 - B \overline{Q_2}$$

and

$$\overline{D}_2 Q_1 = -\overline{A}_2 Q_1 - \overline{B}\,\overline{Q}_1.$$

Denote $\overline{\partial} = \overline{D}_2$, $a = -\overline{A}_2$, $b = -B$, $w = Q_2$, $W = Q_1$, $z = x + y\mathbf{k}$, where $x = x_2$, $y = x_1$ and for convenience we denote $\mathbf{k} = e_3$. Then we reduce the Dirac equation with electromagnetic and scalar potentials independent of x_3 to a pair of Vekua-type equations

$$\overline{\partial} w = a w + b \overline{w} \tag{5}$$

and

$$\overline{\partial} W = a W + \overline{b W}. \tag{6}$$

The difference between the bicomplex equations (5) and (6) and the usual complex Vekua equations is revealed if only w or W can take values equal to bicomplex zero divisors (otherwise equations (5) and (6) can be analysed following Bers–Vekua theory [3, 19]). Let us study this possibility with the aid of the following pair of projection operators:

$$P^+ = \tfrac{1}{2}(1 + \mathrm{i}\mathbf{k}) \qquad \text{and} \qquad P^- = \tfrac{1}{2}(1 - \mathrm{i}\mathbf{k}).$$

The set of bicomplex zero divisors, that is of nonzero elements $q = q_0 + q_1 \mathbf{k}$, $\{q_0, q_1\} \subset \mathbb{C}$ such that

$$q \overline{q} = (q_0 + q_1 \mathbf{k})(q_0 - q_1 \mathbf{k}) = 0, \tag{7}$$

we denote by $\mathfrak{S}$.

Lemma 1. *Let q be a bicomplex number of the form $q = q_0 + q_1 \mathbf{k}$, $\{q_0, q_1\} \subset \mathbb{C}$. If $q \in \mathfrak{S}$ then $q = 2P^+ q_0$ or $q = 2P^- q_0$.*

Proof. From (7) it follows that $q_0^2 + q_1^2 = 0$ which gives us that $q_1 = \pm \mathrm{i} q_0$. That is $q = q_0(1 + \mathrm{i}\mathbf{k})$ or $q = q_0(1 - \mathrm{i}\mathbf{k})$. □

For other results on bicomplex numbers we refer to [16].

Let Ω denote a bounded, simply connected domain in the plane of the variable z.

Theorem 2. *Let $b(z) \notin \mathfrak{S} \cup \{0\}$, $\forall z \in \Omega$ and w, W be solutions of (5) and (6), respectively. Then $w(z) \notin \mathfrak{S}$ and $W(z) \notin \mathfrak{S}$, $\forall z \in \Omega$.*

Proof. Assume that $w(z) \in \mathfrak{S}$ for some $z \in \Omega$. For definiteness, let $w(z) = 2P^+ w_0(z)$. Then from (5) we have

$$\overline{\partial} P^+ w_0 = a P^+ w_0 + b P^- w_0.$$

Applying P^- to this equality we find that $P^- b = 0$ which is a contradiction. □

Thus, if the coefficient b does not have zeros and does not turn into a zero divisor at any point of the domain of interest, the solutions of (5) and (6) will not be zero divisors either and the whole theory of pseudoanalytic functions is applicable without changes to the bicomplex equations (5) and (6). As an example, let us formulate one of the main results of the theory, the similarity principle which is the basic tool for studying the distribution of zeros and of singularities of pseudoanalytic functions as well as boundary value problems [19].

Theorem 3. *Let w be a regular solution of (5) in a domain Ω and let $b(z) \notin \mathfrak{S} \cup \{0\}$, $\forall z \in \Omega$. Then the bicomplex function $\Phi = w \cdot \mathrm{e}^h$, where*

$$h(z) = \frac{1}{2\pi} \int_\Omega \frac{g(\tau)\,\mathrm{d}\tau}{\tau - z},$$

$$g(z) = \begin{cases} a(z) + b(z)\dfrac{\overline{w}(z)}{w(z)} & \text{if} \quad w(z) \neq 0, \quad z \in \Omega, \\ a(z) + b(z) & \text{if} \quad w(z) = 0, \quad z \in \Omega \end{cases}$$

is a solution of the equation $\overline{\partial}\Phi = 0$ in Ω.

The proof of this theorem is completely analogous to that given in [19]. It would be interesting to extend this result to the case of b being a zero divisor in the whole domain Ω or in some points.

This theorem opens the way to generalize many classical results from theory of analytic functions to the case of solutions of equations (5) and (6) by analogy with [19]. Nevertheless, in the present work, we prefer to explore another possibility. Namely, we show how the application of Bers' theory of pseudoanalytic functions allows us to obtain explicitly a complete system of solutions of the Dirac equation with a scalar potential depending on one variable.

5. Some definitions and results from Bers' theory for bicomplex pseudoanalytic functions

5.1. Generating pair, derivative and antiderivative

Following [3], we introduce the notion of a bicomplex generating pair.

Definition 4. *A pair of bicomplex functions $F = F_0 + F_1\mathbf{k}$ and $G = G_0 + G_1\mathbf{k}$, possessing in Ω partial derivatives with respect to the real variables x and y, is said to be a generating pair if it satisfies the inequality*

$$\operatorname{Vec}(\overline{F}G) \neq 0 \qquad \text{in} \quad \Omega.$$

The following expressions are called characteristic coefficients of the pair (F, G)

$$a_{(F,G)} = -\frac{\overline{F}G_{\overline{z}} - F_{\overline{z}}\overline{G}}{F\overline{G} - \overline{F}G}, \qquad b_{(F,G)} = \frac{FG_{\overline{z}} - F_{\overline{z}}G}{F\overline{G} - \overline{F}G},$$

$$A_{(F,G)} = -\frac{\overline{F}G_{\overline{z}} - F_{\overline{z}}\overline{G}}{F\overline{G} - \overline{F}G}, \qquad B_{(F,G)} = \frac{FG_{\overline{z}} - F_{\overline{z}}G}{F\overline{G} - \overline{F}G},$$

where the subindex $\overline{z}$ or z means the application of $\overline{\partial}$ or ∂, respectively.

Every bicomplex function W defined in a subdomain of Ω admits the unique representation $W = \phi F + \psi G$ where the functions ϕ and ψ are complex valued.

The (F, G)-derivative $\dot{W} = \frac{\mathrm{d}_{(F,G)}W}{\mathrm{d}z}$ of a function W exists and has the form

$$\dot{W} = \phi_z F + \psi_z G = W_z - A_{(F,G)}W - B_{(F,G)}\overline{W} \tag{8}$$

if and only if

$$\phi_{\overline{z}} F + \psi_{\overline{z}} G = 0. \tag{9}$$

This last equation can be rewritten in the following form:

$$W_{\overline{z}} = a_{(F,G)}W + b_{(F,G)}\overline{W}$$

which we call the bicomplex Vekua equation. Solutions of this equation are called (F, G)-pseudoanalytic functions.

Remark 5. The functions F and G are (F, G)-pseudoanalytic and $\dot{F} \equiv \dot{G} \equiv 0$.

Definition 6. *Let (F, G) and (F_1, G_1) be two generating pairs in Ω. (F_1, G_1) is called successor of (F, G) and (F, G) is called predecessor of (F_1, G_1) if*

$$a_{(F_1,G_1)} = a_{(F,G)} \qquad \text{and} \qquad b_{(F_1,G_1)} = -B_{(F,G)}.$$

The importance of this definition becomes obvious from the following statement.

Theorem 7. *Let W be an (F, G)-pseudoanalytic function and let (F_1, G_1) be a successor of (F, G). Then $\dot{W}$ is an (F_1, G_1)-pseudoanalytic function.*

Definition 8. *Let (F, G) be a generating pair. Its adjoint generating pair $(F, G)^* = (F^*, G^*)$ is defined by the formulae*

$$F^* = -\frac{2\overline{F}}{F\overline{G} - \overline{F}G}, \qquad G^* = \frac{2\overline{G}}{F\overline{G} - \overline{F}G}.$$

The (F, G)-integral is defined as follows:

$$\int_\Gamma W \,\mathrm{d}_{(F,G)}z = \frac{1}{2}\left(F(z_1)\,\mathrm{Sc}\int_\Gamma G^* W\,\mathrm{d}z + G(z_1)\,\mathrm{Sc}\int_\Gamma F^* W\,\mathrm{d}z\right)$$

where Γ is a rectifiable curve leading from z_0 to z_1.

If $W = \phi F + \psi G$ is an (F, G)-pseudoanalytic function where ϕ and ψ are complex-valued functions then

$$\int_{z_0}^{z} \dot{W}\,\mathrm{d}_{(F,G)}z = W(z) - \phi(z_0)F(z) - \psi(z_0)G(z), \tag{10}$$

and as $\dot{F} = \dot{G} = 0$, this integral is path independent and represents the (F, G)-antiderivative of $\dot{W}$.

5.2. Generating sequences and Taylor series in formal powers

Definition 9. *A sequence of generating pairs* $\{(F_m, G_m)\}$, $m = 0, \pm 1, \pm 2, \ldots,$ *is called a generating sequence if* (F_{m+1}, G_{m+1}) *is a successor of* (F_m, G_m)*. If* $(F_0, G_0) = (F, G)$*, we say that* (F, G) *is embedded in* $\{(F_m, G_m)\}$.

Theorem 10. *Let* (F, G) *be a generating pair in* Ω*. Let* Ω_1 *be a bounded domain,* $\overline{\Omega}_1 \subset \Omega$*. Then,* (F, G) *can be embedded in a generating sequence in* Ω_1.

Definition 11. *A generating sequence* $\{(F_m, G_m)\}$ *is said to have period* $\mu > 0$ *if* $(F_{m+\mu}, G_{m+\mu})$ *is equivalent to* (F_m, G_m) *that is their characteristic coefficients coincide.*

Let W be an (F, G)-pseudoanalytic function. Using a generating sequence in which (F, G) is embedded we can define the higher derivatives of W by the recursion formula

$$W^{[0]} = W; \qquad W^{[m+1]} = \frac{\mathrm{d}_{(F_m,G_m)} W^{[m]}}{\mathrm{d}z}, \qquad m = 1, 2, \ldots.$$

Definition 12. *The formal power* $Z_m^{(0)}(a, z_0; z)$ *with centre at* $z_0 \in \Omega$*, coefficient* a *and exponent* 0 *is defined as the linear combination of the generators* F_m, G_m *with complex constant coefficients* λ, μ *chosen so that* $\lambda F_m(z_0) + \mu G_m(z_0) = a$*. The formal powers with exponents* $n = 1, 2, \ldots$ *are defined by the recursion formula*

$$Z_m^{(n+1)}(a, z_0; z) = (n+1) \int_{z_0}^{z} Z_{m+1}^{(n)}(a, z_0; \zeta)\, \mathrm{d}_{(F_m,G_m)}\zeta. \tag{11}$$

This definition implies the following properties:

(1) $Z_m^{(n)}(a, z_0; z)$ is an (F_m, G_m)-pseudoanalytic function of z.

(2) If a' and a'' are complex constants, then

$$Z_m^{(n)}(a' + \mathbf{k}a'', z_0; z) = a' Z_m^{(n)}(1, z_0; z) + a'' Z_m^{(n)}(\mathbf{k}, z_0; z).$$

(3) The formal powers satisfy the differential relations

$$\frac{\mathrm{d}_{(F_m,G_m)} Z_m^{(n)}(a, z_0; z)}{\mathrm{d}z} = n Z_{m+1}^{(n-1)}(a, z_0; z).$$

(4) The asymptotic formulae

$$Z_m^{(n)}(a, z_0; z) \sim a(z - z_0)^n, \qquad z \to z_0$$

hold.

Assume now that

$$W(z) = \sum_{n=0}^{\infty} Z^{(n)}(a_n, z_0; z) \tag{12}$$

where the absence of the subindex m means that all the formal powers correspond to the same generating pair (F, G), and the series converges uniformly in some neighbourhood of z_0. It can be shown that the uniform limit of pseudoanalytic functions is pseudoanalytic, and that a uniformly convergent series of (F, G)-pseudoanalytic functions can be (F, G)-differentiated term by term. Hence, the function W in (12) is (F, G)-pseudoanalytic and its rth derivative admits the expansion

$$W^{[r]}(z) = \sum_{n=r}^{\infty} n(n-1)\cdots(n-r+1) Z_r^{(n-r)}(a_n, z_0; z).$$

From this, the Taylor formulae for the coefficients are obtained

$$a_n = \frac{W^{[n]}(z_0)}{n!}. \tag{13}$$

Definition 13. *Let $W(z)$ be a given (F, G)-pseudoanalytic function defined for small values of $|z - z_0|$. The series*

$$\sum_{n=0}^{\infty} Z^{(n)}(a_n, z_0; z) \tag{14}$$

with the coefficients given by (13) is called the Taylor series of W at z_0, formed with formal powers.

The Taylor series always represents the function asymptotically:

$$W(z) - \sum_{n=0}^{N} Z^{(n)}(a_n, z_0; z) = O(|z - z_0|^{N+1}), \qquad z \to z_0, \tag{15}$$

for all N. This implies (since a pseudoanalytic function cannot have a zero of arbitrarily high order without vanishing identically) that the sequence of derivatives $\{W^{[n]}(z_0)\}$ determines the function W uniquely.

If the series (14) converges uniformly in a neighbourhood of z_0, it converges to the function W.

Theorem 14. *The formal Taylor expansion (14) of a pseudoanalytic function in formal powers defined by a periodic generating sequence converges in some neighbourhood of the centre.*

6. Special class of Vekua equations

The following important class of Vekua equations was considered in [12]. Let f_0 be a complex valued (with respect to i), twice differentiable nonvanishing function defined on Ω. Consider the equation

$$\overline{\partial} W = \frac{\overline{\partial} f_0}{f_0} \overline{W} \qquad \text{in} \quad \Omega. \tag{16}$$

Denote $\nu_1 = \Delta f_0 / f_0$.

Theorem 15 [12]. *If $W = W_1 + W_2\mathbf{k}$ is a solution of (16) then $W_1 = \mathrm{Sc}W$ is a solution of the stationary Schrödinger equation*

$$-\Delta W_1 + \nu_1 W_1 = 0 \qquad \text{in} \quad \Omega \tag{17}$$

and $W_2 = \mathrm{Vec}\, W$ is a solution of the associated Schrödinger equation

$$-\Delta W_2 + \nu_2 W_2 = 0 \qquad \text{in} \quad \Omega \tag{18}$$

where $\nu_2 = 2(\overline{\partial} f_0 \cdot \partial f_0)/f_0^2 - \nu_1$.

Moreover, in [12], a simple formula was obtained which allows us for any given solution W_1 of (17) to construct such a solution W_2 of (18) that $W = W_1 + W_2\mathbf{k}$ will be a solution of (16) generalizing in this way the well-known procedure for constructing conjugate harmonic functions in complex analysis.

7. Dirac equation with a scalar potential

Let us show that the Dirac equation with a scalar potential depending on one real variable reduces to a bicomplex Vekua equation of the form (16).

Let $p_{\mathrm{sc}} = p(x)$ and $p_{\mathrm{el}} \equiv 0$, $A_k \equiv 0$, $k = 1, 2, 3$. Then according to section 4, the Dirac equation is equivalent to the pair of bicomplex Vekua equations

$$\overline{\partial} w = b\overline{w} \tag{19}$$

and

$$\overline{\partial} W = \overline{bW} \tag{20}$$

with $b = p(x) + m - \mathrm{i}\omega\mathbf{k}$.

Let $f_0 = \exp(P(x) + mx + \mathrm{i}\omega y)$, where P is an antiderivative of p. Then we have

$$\overline{b} = \overline{\partial} f_0 / f_0.$$

Note that due to theorem 15 if the bicomplex function W is a solution of (20) then the complex function $W_1 = \mathrm{Sc}\, W$ is a solution of the stationary Schrödinger equation (17) where

$$\nu_1(x) = p'(x) + (p(x) + m)^2 - \omega^2, \tag{21}$$

and the function $W_2 = \mathrm{Vec}\, W$ is a solution of equation (18) where

$$\nu_2(x) = -p'(x) + (p(x) + m)^2 - \omega^2. \tag{22}$$

Let us note that both Schrödinger equations (17) and (18) in this case admit separation of variables. Nevertheless, this does not imply they can be solved explicitly. In general, this is not the case. However, we will show how using our approach and Bers' theory for both of them one can construct in explicit form a locally complete system of exact solutions.

Consider equation (20). It is easy to see that the pair of functions

$$F = f_0 \qquad \text{and} \qquad G = \frac{\mathbf{k}}{f_0} \tag{23}$$

represents a generating pair for (20). Note that $F = \mathrm{e}^{\sigma}$ and $G = \mathrm{e}^{-\sigma}\mathbf{k}$, where $\sigma = \alpha(x) + \beta(y)$ and $\alpha(x) = P(x) + mx$, $\beta(y) = \mathrm{i}\omega y$. For a generating pair of such special kind, it is easy to construct a successor [3]. Let $\tau = -\alpha(x) + \beta(y)$. Then the pair $F_1 = \mathrm{e}^{\tau}$ and $G_1 = \mathrm{e}^{-\tau}\mathbf{k}$ is a successor of (F, G). Moreover, (F, G) is a successor of (F_1, G_1). Thus, for (F, G) we obtain a complete periodic generating sequence of a period 2 in explicit form (for explicitly constructed, in general, non-periodic generating sequences in a far more general situation we refer to [12]).

The fact that we have a generating sequence in explicit form implies that we are able to construct the corresponding formal powers of any order explicitly and therefore to obtain a locally complete system of exact solutions of the Dirac equation with a scalar potential depending on one variable as well as of the stationary Schrödinger equations (17) and (18) with potentials (21) and (22), respectively.

As a first step, we construct the adjoint generating pair (see definition 8):

$$F^* = -f_0\mathbf{k} \qquad \text{and} \qquad G^* = \frac{1}{f_0}.$$

Next, we write down the expression for the (F, G)-integral:

$$\int_\Gamma W\, \mathrm{d}_{(F,G)}z = \frac{1}{2}\left(f_0(z_1)\, \mathrm{Sc} \int_\Gamma \frac{W(z)}{f_0(z)}\, \mathrm{d}z - \frac{\mathbf{k}}{f_0(z_1)}\, \mathrm{Sc} \int_\Gamma f_0(z) W(z) \mathbf{k}\, \mathrm{d}z \right).$$

By definition, the formal power $Z^{(0)}(a, z_0; z)$ for equation (20) has the form

$$Z^{(0)}(a, z_0; z) = \lambda F(z) + \mu G(z),$$

where the complex constants λ and μ are chosen so that $\lambda F(z_0) + \mu G(z_0) = a$. That is,

$$Z^{(0)}(a, z_0; z) = \lambda \exp(P(x) + mx + \mathrm{i}\omega y) + \mu \exp(-(P(x) + mx + \mathrm{i}\omega y))\mathbf{k}.$$

In order to obtain $Z^{(1)}(a, z_0; z)$ we should take the (F, G)-integral of $Z_1^{(0)}(a, z_0; z)$, where

$$Z_1^{(0)}(a, z_0; z) = \lambda_1 F_1(z) + \mu_1 G_1(z),$$

with $\lambda_1 F_1(z_0) + \mu_1 G_1(z_0) = a$. Thus,

$$\begin{aligned}
Z^{(1)}(a, z_0; z) &= \int_{z_0}^{z} (\lambda_1 F_1(\zeta) + \mu_1 G_1(\zeta))\, \mathrm{d}_{(F,G)}\zeta \\
&= \frac{1}{2}\Bigg\{ \exp(P(x) + mx + \mathrm{i}\omega y)\, \mathrm{Sc} \int_{z_0}^{z} \exp(-P(x') - mx' - \mathrm{i}\omega y') \\
&\quad \times (\lambda_1 \exp(-P(x') - mx' + \mathrm{i}\omega y') + \mu_1 \exp(P(x') + mx' - \mathrm{i}\omega y')\mathbf{k})\, \mathrm{d}\zeta \\
&\quad - \exp(-P(x) - mx - \mathrm{i}\omega y)\mathbf{k}\, \mathrm{Sc} \int_{z_0}^{z} \exp(P(x') + mx' + \mathrm{i}\omega y')\mathbf{k} \\
&\quad \times (\lambda_1 \exp(-P(x') - mx' + \mathrm{i}\omega y') + \mu_1 \exp(P(x') + mx' - \mathrm{i}\omega y')\mathbf{k})\, \mathrm{d}\zeta \Bigg\} \\
&= \frac{1}{2}\Bigg\{ \exp(P(x) + mx + \mathrm{i}\omega y)\, \mathrm{Sc} \int_{z_0}^{z} (\lambda_1 \exp(-2(P(x') + mx')) + \mu_1 \mathrm{e}^{-2\mathrm{i}\omega y'}\mathbf{k})\, \mathrm{d}\zeta \\
&\quad - \exp(-P(x) - mx - \mathrm{i}\omega y)\mathbf{k}\, \mathrm{Sc} \int_{z_0}^{z} (\lambda_1 \mathrm{e}^{2\mathrm{i}\omega y'}\mathbf{k} - \mu_1 \exp(2(P(x') + mx')))\, \mathrm{d}\zeta \Bigg\}
\end{aligned}$$

where $\zeta = x' + y'\mathbf{k}$.

For $Z^{(2)}(a, z_0; z)$, by definition 12, we have

$$Z^{(2)}(a, z_0; z) = 2 \int_{z_0}^{z} Z_1^{(1)}(a, z_0; \zeta)\, \mathrm{d}_{(F,G)}\zeta, \tag{24}$$

where $Z_1^{(1)}(a, z_0; \zeta)$ in its turn can be found from the equality

$$Z_1^{(1)}(a, z_0; z) = \int_{z_0}^{z} Z_2^{(0)}(a, z_0; \zeta)\, \mathrm{d}_{(F_1,G_1)}\zeta. \tag{25}$$

We note that due to periodicity of the generating sequence containing the generating pair (23),

$$Z_2^{(0)}(a, z_0; \zeta) = Z^{(0)}(a, z_0; \zeta).$$

The adjoint pair for (F_1, G_1) necessary for the (F_1, G_1)-integral in (25) has the form

$$F_1^* = -\mathrm{e}^{\tau}\mathbf{k} \qquad \text{and} \qquad G_1^* = \mathrm{e}^{-\tau}.$$

Thus,

$$\begin{aligned}
Z_1^{(1)}(a, z_0; z) &= \frac{1}{2}\Bigg\{ \exp(-P(x) - mx + \mathrm{i}\omega y) \\
&\quad \times \mathrm{Sc} \int_{z_0}^{z} \exp(P(x') + mx' - \mathrm{i}\omega y')(\lambda \exp(P(x') + mx' + \mathrm{i}\omega y') \\
&\quad + \mu \exp(-P(x') - mx' - \mathrm{i}\omega y')\mathbf{k})\, \mathrm{d}\zeta - \exp(P(x) + mx - \mathrm{i}\omega y)\mathbf{k} \\
&\quad \times \mathrm{Sc} \int_{z_0}^{z} \exp(-P(x') - mx' + \mathrm{i}\omega y')\mathbf{k}(\lambda \exp(P(x') + mx' + \mathrm{i}\omega y')
\end{aligned}$$

$$+\mu\exp(-P(x')-mx'-\mathrm{i}\omega y')\mathbf{k})\,\mathrm{d}\zeta\Bigg\}$$

$$=\frac{1}{2}\Bigg\{\exp(-P(x)-mx+\mathrm{i}\omega y)\,\mathrm{Sc}\int_{z_0}^{z}(\lambda\exp(2(P(x')+mx'))$$

$$+\mu\mathrm{e}^{-2\mathrm{i}\omega y'}\mathbf{k})\,\mathrm{d}\zeta-\exp(P(x)+mx-\mathrm{i}\omega y)\mathbf{k}$$

$$\times\,\mathrm{Sc}\int_{z_0}^{z}(\lambda\mathrm{e}^{2\mathrm{i}\omega y'}\mathbf{k}-\mu\exp(-2(P(x')+mx')))\,\mathrm{d}\zeta\Bigg\}.$$

Substitution of this expression into (24) gives us the formal power $Z^{(2)}(a, z_0; z)$, and this algorithmically simple procedure can be continued indefinitely. As a result, we obtain an infinite system of formal powers which at least locally gives us a complete system of solutions of (20) in the sense that any regular solution of (20) can be approximated arbitrarily closely by a finite linear combination of formal powers (formula (15)). Moreover, as the corresponding generating sequence is periodic, theorem 14 is valid, and therefore we can guarantee the convergence of a Taylor expansion in the formal powers to a corresponding solution of (20) in some neighbourhood of z_0.

A similar procedure also works for equation (19). Note that the pair of functions $F_1\mathbf{k} = \mathrm{e}^{\tau}\mathbf{k}$ and $G_1\mathbf{k} = -\mathrm{e}^{-\tau}$ is a generating pair corresponding to (19).

As any solution of the Schrödinger equation (17) with the potential ν_1 defined by (21) is the scalar part of some solution of (20) and any solution of (18) with the potential (22) is the vector part of some solution of (20), the scalar and the vector parts of the constructed system of formal powers give us locally complete systems of solutions of (17) and (18), respectively.

This last result can also be interpreted in the following way. Consider the equation

$$-\Delta f+\nu f=\omega^2 f \qquad \text{in} \quad \Omega \tag{26}$$

where f is a complex twice continuously differentiable function of two real variables x and y, and ν is a complex-valued function of one real variable x, ω is a complex constant. Suppose we are given a particular solution $f_0 = f_0(x)$ of the ordinary differential equation

$$-\frac{\mathrm{d}^2 f_0}{\mathrm{d}x^2}+\nu f_0=0. \tag{27}$$

This implies that we are able to represent ν in the form $\nu = p' + p^2$ where $p = f_0'/f_0$. Then we observe that (26) is precisely equation (17) with $m = 0$ in (21). Thus, our result means that if we are able to solve the ordinary differential equation (27) then we can construct explicitly a locally complete system of exact solutions to (26) for any ω. For this one should consider the bicomplex Vekua equation (20) and follow the procedure described above for constructing the corresponding system of formal powers. Then the scalar part of the system gives us a locally complete system of exact solutions to (26).

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